

Rigidity of the L^p -norm of the Poisson bracket on surfaces

Karina Samvelyan* and Frol Zapolsky†

September 29, 2016

Abstract

For a symplectic manifold (M, ω) let $\{\cdot, \cdot\}$ be the corresponding Poisson bracket. In this note we prove that the functional

$$(F, G) \mapsto \|\{F, G\}\|_{L^p(M)}$$

is lower-semicontinuous with respect to the C^0 -norm on $C_c^\infty(M)$ when $\dim M = 2$ and $p < \infty$, extending previous rigidity results for $p = \infty$ in arbitrary dimension.

1 Introduction and main result

One of the fascinating manifestations of rigidity in symplectic topology is the unexpected robust behavior of the Poisson bracket with respect to the C^0 -norm on the space of smooth functions, discovered by Cardin–Viterbo [CV08]. To state their seminal result, let (M, ω) be a symplectic manifold without boundary, and let us endow the space $C_c^\infty(M)$ of smooth compactly supported functions on M with the topology induced by the supremum norm $\|\cdot\|_{C^0}$. We write $\xrightarrow{C^0}$ to indicate convergence with respect to this topology.

The Poisson bracket of $F, G \in C^\infty(M)$ is the function

$$\{F, G\} = -\omega(X_F, X_G) = dF(X_G),$$

where for $H \in C^\infty(M)$ its Hamiltonian vector field X_H is defined by $\omega(X_H, \cdot) = -dH$.

*School of Mathematical Sciences, Faculty of Exact Sciences, Tel Aviv University, karina.samvelyan@gmail.com.

†Department of Mathematics, Faculty of Natural Sciences, University of Haifa, frol.zapolsky@gmail.com.

Theorem 1.1 (Cardin–Viterbo [CV08]). *Let N be \mathbb{R}^n or a closed manifold,¹⁾ and assume that $M = T^*N$ and ω is the canonical symplectic form. Let $F, G \in C_c^\infty(M)$ be such that $\{F, G\} \neq 0$. Then*

$$\liminf_{\overline{F} \xrightarrow{C^0}, \overline{G} \xrightarrow{C^0} F} \|\{\overline{F}, \overline{G}\}\|_{C^0} > 0.$$

This means that if two functions do not Poisson commute, it is impossible to approximate them, in the C^0 sense, by Poisson commuting, or even asymptotically commuting, functions. This behavior is surprising because the Poisson bracket is defined in terms of the first derivatives of the functions and thus *a priori* it is unknown how it changes under C^0 perturbations. For surfaces, a stronger form of this statement was proved in [Zap07]:²⁾

Theorem 1.2. *Assume $\dim M = 2$. Then for $F, G \in C_c^\infty(M)$ the functional $\|\{\cdot, \cdot\}\|_{C^0}$ is lower-semicontinuous with respect to C^0 -norm, meaning*

$$\liminf_{\overline{F} \xrightarrow{C^0}, \overline{G} \xrightarrow{C^0} G} \|\{\overline{F}, \overline{G}\}\|_{C^0} = \|\{F, G\}\|_{C^0}.$$

This result was proved using methods of classical analysis in dimension two. It was later generalized, using methods of “hard” symplectic topology, including the Hofer metric and the energy-capacity inequality, to arbitrary dimension:

Theorem 1.3 ([EP10], [Buh10]). *For M of arbitrary dimension and any $F, G \in C_c^\infty(M)$ we have*

$$\liminf_{\overline{F} \xrightarrow{C^0} F, \overline{G} \xrightarrow{C^0} G} \|\{\overline{F}, \overline{G}\}\|_{C^0} = \|\{F, G\}\|_{C^0}.$$

Our main result in this note is the following rigidity phenomenon in dimension two, proved using a refinement of the technique from [Zap07]:

Theorem 1.4. *Assume $\dim M = 2$ and $p \in [1, \infty)$. Then for $F, G \in C_c^\infty(M)$ we have*

$$\liminf_{\overline{F} \xrightarrow{C^0} F, \overline{G} \xrightarrow{C^0} G} \|\{\overline{F}, \overline{G}\}\|_{L^p(M)} = \|\{F, G\}\|_{L^p(M)}.$$

Here and in the rest of the note we denote by $\|H\|_{L^p(X)}$ the L^p -norm, with respect to the measure induced by ω , of a function H defined on a measurable subset $X \subset M$.

Whether this behavior persists in higher dimension is currently unknown. Therefore we ask the following question.

¹⁾The proof uses generating functions for Lagrangians in T^*N , therefore it is plausible that it extends to more general N , however this formulation suffices to illustrate the main point.

²⁾See also [EPZ07] for intermediate quantitative results in arbitrary dimension using symplectic quasi-states.

Question 1.5. Is the functional $(F, G) \mapsto \|\{F, G\}\|_{L^p(M)}$ lower semi-continuous for M of arbitrary dimension and finite p ?

In view of the results in [Buh10], it is also natural to ask the following.

Question 1.6. What is the modulus of semi-continuity of the functional $(F, G) \mapsto \|\{F, G\}\|_{L^p(M)}$? Is there a constant $\kappa > 0$ such that

$$\inf_{\|\bar{F}-F\|_{C^0}, \|\bar{G}-G\|_{C^0} \leq \delta} \|\{\bar{F}, \bar{G}\}\|_{L^p(M)} \geq \|\{F, G\}\|_{L^p(M)} - \text{const}(F, G) \cdot \delta^\kappa?$$

The rigid behavior with respect to the C^0 -norm should be contrasted with the following result.

Theorem 1.7 ([Sam15]). *Let M have arbitrary dimension $2n$, let $q \in [1, \infty)$, and let $F, G \in C_c^\infty(M)$. Then for any $\varepsilon > 0$ and a compact submanifold with boundary $C \subset M$ of dimension $2n$, whose interior contains $\text{supp } F \cup \text{supp } G$, there exist $\tilde{F}, \tilde{G} \in C_c^\infty(M)$ supported in C , such that*

$$\|\tilde{F} - F\|_{C^0} < \varepsilon, \quad \|\tilde{G} - G\|_{L^q(M)} < \varepsilon^{1/q}, \quad \text{and} \quad \{\tilde{F}, \tilde{G}\} \equiv 0.$$

In particular, for any $p \in [1, \infty]$,

$$\liminf_{\tilde{F} \xrightarrow{C^0} F, \tilde{G} \xrightarrow{L^q} G} \|\{\tilde{F}, \tilde{G}\}\|_{L^p(M)} = \liminf_{\tilde{F} \xrightarrow{L^q} F, \tilde{G} \xrightarrow{L^q} G} \|\{\tilde{F}, \tilde{G}\}\|_{L^p(M)} = 0.$$

Here we use the fact that $\|H\|_{L^q(M)} \leq (\int_C \omega^n)^{1/q} \cdot \|H\|_{C^0}$ for $H \in C^\infty(M)$ with $\text{supp } H \subset C$. This means that the L^p -norm of the Poisson bracket becomes flexible if we take the L^q -topology on $C_c^\infty(M)$ for finite q .

For the sake of completeness, we provide a proof of the theorem in the next section.

Remark 1.8. Note that for continuous functions the L^∞ -norm and the C^0 -norm coincide.

Acknowledgements. We wish to thank Lev Buhovsky and Leonid Polterovich for reading a preliminary version of the paper and making useful comments, and for their interest. KS is partially supported by the Israel Science Foundation grant number 178/13, and by the European Research Council Advanced grant number 338809. FZ is partially supported by grant number 1281 from the GIF, the German-Israeli Foundation for Scientific Research and Development, and by grant number 1825/14 from the Israel Science Foundation.

2 Proofs

Proof (of Theorem 1.4). Let us give an overview of the proof before passing to the details. The actual logical order of the proof is somewhat different from this summary.

We define the map

$$\Phi: M \rightarrow \mathbb{R}^2 \quad \text{by} \quad \Phi(z) = (F(z), G(z)).$$

The main point is that since $\dim M = 2$, the Poisson bracket $\{F, G\}$ is related to Φ via

$$\Phi^*(dx \wedge dy) = dF \wedge dG = -\{F, G\}\omega,$$

where (x, y) are the coordinate functions on \mathbb{R}^2 . We see that a point $z \in M$ is regular for Φ if and only if $\{F, G\}(z) \neq 0$. We let $U \subset \mathbb{R}^2$ be the set of regular values of Φ in $\text{im } \Phi$.

Consider now the subset $K_n \subset U$ comprised of squares of size $\frac{1}{n}$ with vertices in the grid $\frac{1}{n}\mathbb{Z} \times \frac{1}{n}\mathbb{Z}$ with $n \in \mathbb{N}$ large so that $\|\{F, G\}\|_{L^p(\Phi^{-1}(K_n))}$ is close to $\|\{F, G\}\|_{L^p(M)}$. Next we *subdivide each square in K_n into squares of size $\frac{1}{kn}$, $k \in \mathbb{N}$* . For such a square Q , a connected component $Q' \subset \Phi^{-1}(Q)$, and k large, the *oscillation of $\{F, G\}$ over Q' can be made arbitrarily small* for all such Q' , which allows us to relate the L^1 - and the L^p -norms of $\{F, G\}$ over Q' . Then we use the *lower semi-continuity of the L^1 -norm* to pass to $\|\overline{\{F, G\}}\|_{L^1(Q')}$. Finally the *Hölder inequality* brings us back to $\|\overline{\{F, G\}}\|_{L^p(Q')}$.

Remark 2.1. We wish to note here that the use of two scales, $\frac{1}{n}$ and $\frac{1}{kn}$, seems to stem from convenience rather than being a reflection of something deeper. We must simultaneously approximate the L^p -norm of $\{F, G\}$ and control its oscillation, and this double subdivision is a way to do it.

We now give the details of the proof.

Remark 2.2. Since M is assumed to have no boundary, and $\{F, G\}$ has compact support, Φ is a covering map over U . In particular, the lifting property of a covering implies that if $Y \subset U$ is a path-connected simply connected subset, then $\Phi|_{\Phi^{-1}(Y)}: \Phi^{-1}(Y) \rightarrow Y$ is a trivial covering, that is $\Phi^{-1}(Y)$ is a disjoint union of path components, each one projected homeomorphically onto Y by Φ . If Y is in addition a submanifold with corners, then, since Φ is smooth, these components are themselves submanifolds with corners, projected in fact diffeomorphically onto Y .

For $n \in \mathbb{N}$ let $K_n \subset \mathbb{R}^2$ be the union of squares of the form $[\frac{i}{n}, \frac{i+1}{n}] \times [\frac{j}{n}, \frac{j+1}{n}]$, where $i, j \in \mathbb{Z}$, contained in U . For $k \in \mathbb{N}$ consider a square $Q = [\frac{i}{kn}, \frac{i+1}{kn}] \times [\frac{j}{kn}, \frac{j+1}{kn}]$, where $i, j \in \mathbb{Z}$, and assume it is contained in K_n .

By Remark 2.2, $\Phi^{-1}(Q)$ is a disjoint union of connected components, each of which is mapped by Φ diffeomorphically onto Q . See fig. 1. Let $\mathcal{Q}_{n,k}$ be the collection of all such connected components for all such Q . The next lemma states that the oscillation of $|\{F, G\}|^p$ over the sets in $\mathcal{Q}_{n,k}$ can be made arbitrarily small as $k \rightarrow \infty$.

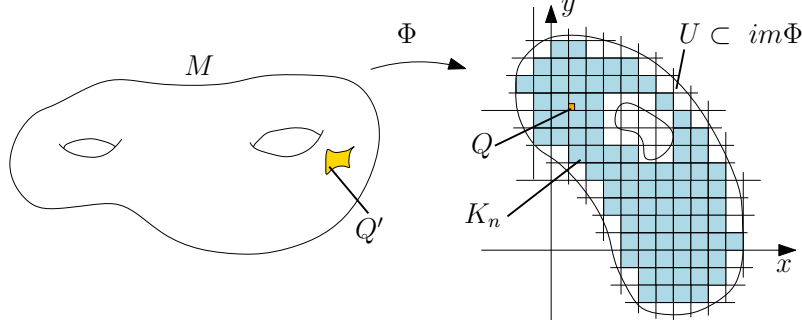


Figure 1: The set $K_n \subset U$ and an element $\Phi(Q') = Q$ in its subdivision.

Lemma 2.3. $\lim_{k \rightarrow \infty} \max_{Q' \in \mathcal{Q}_{n,k}} \text{osc}_{Q'} |\{F, G\}|^p = 0$.

Fix $\varepsilon > 0$ and let $k \in \mathbb{N}$ be such that $\max_{Q' \in \mathcal{Q}_{n,k}} \text{osc}_{Q'} |\{F, G\}|^p \leq \varepsilon$. Pick $Q' \in \mathcal{Q}_{n,k}$, let $Q = \Phi(Q') \subset \mathbb{R}^2$, and let $i, j \in \mathbb{Z}$ be such that $Q = [\frac{i}{kn}, \frac{i+1}{kn}] \times [\frac{j}{kn}, \frac{j+1}{kn}]$. For $\delta \in (0, \frac{1}{2kn})$ denote $Q_\delta = [\frac{i}{kn} + \delta, \frac{i+1}{kn} - \delta] \times [\frac{j}{kn} + \delta, \frac{j+1}{kn} - \delta]$. The following lemma is a quantitative local surjectivity result for C^0 -perturbations of Φ . Its proof is an almost verbatim repetition of the one of Lemma 3.1 in [Zap07] and is omitted.

Lemma 2.4. *Let $\delta \in (0, \frac{1}{2kn})$ and let $\overline{F}, \overline{G} \in C_c^\infty(M)$ be such that*

$$\|\overline{F} - F\|_{C^0} \leq \delta, \quad \|\overline{G} - G\|_{C^0} \leq \delta.$$

Define

$$\overline{\Phi}: M \rightarrow \mathbb{R}^2 \quad \text{by} \quad \overline{\Phi}(z) = (\overline{F}(z), \overline{G}(z)).$$

Then we have

$$\overline{\Phi}(Q') \supset Q_\delta. \quad \square$$

Fix $\delta \in (0, \frac{1}{2kn})$ and $\overline{F}, \overline{G} \in C_c^\infty(M)$ with $\|\overline{F} - F\|_{C^0} \leq \delta, \|\overline{G} - G\|_{C^0} \leq \delta$. Let q be such that $1/p + 1/q = 1$. The Hölder inequality allows us to relate the L^p - and the L^1 -norms of $\{\overline{F}, \overline{G}\}$:

$$\|\{\overline{F}, \overline{G}\}\|_{L^p(Q')}^p \geq \|\{\overline{F}, \overline{G}\}\|_{L^1(Q')}^p \|1\|_{L^q(Q')}^{-p}.$$

Let us define the function

$$n_{\overline{\Phi}}: \mathbb{R}^2 \rightarrow \mathbb{N} \cup \{0, \infty\},$$

where $n_{\overline{\Phi}}(u)$ is the number of preimages of u by the restriction of $\overline{\Phi}$ to Q' . Note that by Lemma 2.4 $n_{\overline{\Phi}}(u) \geq 1$ for every $u \in Q_\delta$. The so-called area formula from geometric measure theory [Fed69, Theorem 3.2.3] implies in our case the following identity:

$$\int_{Q'} |d\overline{F} \wedge d\overline{G}| = \int_{\mathbb{R}^2} n_{\overline{\Phi}} dx \wedge dy,$$

where for a 2-form β on M we let $|\beta|$ denote the corresponding density.³⁾

Next, we relate the L^1 -norms of $\{\overline{F}, \overline{G}\}$ and $\{F, G\}$ over Q' :

$$\begin{aligned} \|\{\overline{F}, \overline{G}\}\|_{L^1(Q')} &= \int_{Q'} |d\overline{F} \wedge d\overline{G}| && \text{by the definition of } \{\cdot, \cdot\} \\ &= \int_{\mathbb{R}^2} n_{\overline{\Phi}} dx \wedge dy && \text{by the area formula} \\ &\geq \int_{Q_\delta} dx \wedge dy && \text{since } n_{\overline{\Phi}}|_{Q_\delta} \geq 1. \end{aligned}$$

The last integral is the area of Q_δ , which equals

$$\left(\frac{1}{kn} - 2\delta\right)^2 = (1 - 2kn\delta)^2 \text{area}(Q) = (1 - 2kn\delta)^2 \int_Q dx \wedge dy.$$

We continue:

$$\begin{aligned} \|\{\overline{F}, \overline{G}\}\|_{L^1(Q')} &\geq (1 - 2kn\delta)^2 \int_Q dx \wedge dy \\ &= (1 - 2kn\delta)^2 \int_{\Phi(Q')} |dx \wedge dy| \\ &= (1 - 2kn\delta)^2 \int_{Q'} |\Phi^*(dx \wedge dy)| \\ &= (1 - 2kn\delta)^2 \int_{Q'} |dF \wedge dG| \\ &= (1 - 2kn\delta)^2 \|\{F, G\}\|_{L^1(Q')}, \end{aligned}$$

therefore

$$\|\{\overline{F}, \overline{G}\}\|_{L^1(Q')}^p \geq (1 - 2kn\delta)^{2p} \|\{F, G\}\|_{L^1(Q')}^p.$$

Now we relate the L^1 - and the L^p -norms of $\{F, G\}$ over Q' . Namely, since $\text{osc}_{Q'} |\{F, G\}|^p \leq \varepsilon$, we have

$$\begin{aligned} \|\{F, G\}\|_{L^1(Q')}^p &\geq (\min_{Q'} |\{F, G\}| \int_{Q'} \omega)^p = \min_{Q'} |\{F, G\}|^p (\int_{Q'} \omega)^p \geq \\ &\geq (\max_{Q'} |\{F, G\}|^p - \varepsilon) (\int_{Q'} \omega)^p, \end{aligned}$$

³⁾This can be thought of as the nonnegative measure induced by β ; if $f \in C^\infty(M)$ is such that $\beta = f\omega$, then $\int |\beta| \equiv \int |f|\omega$.

therefore, since $\|1\|_{L^q(Q')}^{-p} = (\int_{Q'} \omega)^{-p/q}$:

$$\|\{F, G\}\|_{L^1(Q')}^p \|1\|_{L^q(Q')}^{-p} \geq (\max_{Q'} |\{F, G\}|^p - \varepsilon) (\int_{Q'} \omega)^{p-p/q}.$$

Since $p - p/q = 1$, we obtain

$$\max_{Q'} |\{F, G\}|^p (\int_{Q'} \omega)^{p-p/q} = \max_{Q'} |\{F, G\}|^p \int_{Q'} \omega \geq \|\{F, G\}\|_{L^p(Q')}^p,$$

thus in total

$$\|\{F, G\}\|_{L^1(Q')}^p \|1\|_{L^q(Q')}^{-p} \geq \|\{F, G\}\|_{L^p(Q')}^p - \varepsilon \int_{Q'} \omega.$$

Assembling all of the above, we obtain the main estimate

$$\|\{\overline{F}, \overline{G}\}\|_{L^p(Q')}^p \geq (1 - 2kn\delta)^{2p} (\|\{F, G\}\|_{L^p(Q')}^p - \varepsilon \int_{Q'} \omega). \quad (1)$$

Note that $\Phi^{-1}(K_n)$ is the essentially disjoint⁴⁾ union of the sets $Q' \in \mathcal{Q}_{n,k}$, and that $\|\cdot\|_{L^p}^p$ is additive with respect to essentially disjoint unions. Thus we have

$$\begin{aligned} \|\{\overline{F}, \overline{G}\}\|_{L^p(M)}^p &\geq \|\{\overline{F}, \overline{G}\}\|_{L^p(\Phi^{-1}(K_n))}^p \\ &= \sum_{Q' \in \mathcal{Q}_{n,k}} \|\{\overline{F}, \overline{G}\}\|_{L^p(Q')}^p \\ &\stackrel{*}{\geq} (1 - 2kn\delta)^{2p} \sum_{Q' \in \mathcal{Q}_{n,k}} (\|\{F, G\}\|_{L^p(Q')}^p - \varepsilon \int_{Q'} \omega) \\ &= (1 - 2kn\delta)^{2p} (\|\{F, G\}\|_{L^p(\Phi^{-1}(K_n))}^p - \varepsilon \int_{\Phi^{-1}(K_n)} \omega) \\ &\geq (1 - 2kn\delta)^{2p} (\|\{F, G\}\|_{L^p(\Phi^{-1}(K_n))}^p - \varepsilon \cdot \text{area supp}\{F, G\}), \end{aligned}$$

where for $\stackrel{*}{\geq}$ we used the main estimate (1), and in the last inequality we used $\Phi^{-1}(K_n) \subset \text{supp}\{F, G\}$. Taking $\delta \rightarrow 0$, we see that

$$\liminf_{\overline{F} \xrightarrow{C^0} F, \overline{G} \xrightarrow{C^0} G} \|\{\overline{F}, \overline{G}\}\|_{L^p(M)}^p \geq \|\{F, G\}\|_{L^p(\Phi^{-1}(K_n))}^p - \varepsilon \cdot \text{area supp}\{F, G\},$$

and since ε was arbitrary, we have

$$\liminf_{\overline{F} \xrightarrow{C^0} F, \overline{G} \xrightarrow{C^0} G} \|\{\overline{F}, \overline{G}\}\|_{L^p(M)}^p \geq \|\{F, G\}\|_{L^p(\Phi^{-1}(K_n))}^p.$$

It remains to invoke the following lemma, which says that the L^p -norm of $\{F, G\}$ can be approximated by looking at the sets $\Phi^{-1}(K_n)$.

⁴⁾ A countable union of subsets is essentially disjoint if the intersection of every two subsets has measure zero.

Lemma 2.5. $\sup_{n \in \mathbb{N}} \|\{F, G\}\|_{L^p(\Phi^{-1}(K_n))}^p = \|\{F, G\}\|_{L^p(M)}^p.$

The proof is thus finished, assuming Lemmas 2.3, 2.5. \square

It remains to prove the lemmas. We keep the notations introduced during the proof of Theorem 1.4.

Proof (of Lemma 2.3). Let $C \subset K_n$ be a square entering the definition of K_n . By Remark 2.2, $\Phi^{-1}(C)$ is a disjoint union of a finite number of components, each projecting diffeomorphically onto C by Φ . Let \mathcal{C} be the collection of all such connected components for all the squares $C \subset K_n$. Note that \mathcal{C} is finite. Pick $C' \in \mathcal{C}$, let $C = \Phi(C')$, and let $P_{C'}: C \rightarrow \mathbb{R}$ be the function $|\{F, G\}|^p \circ (\Phi|_{C'})^{-1}$. Since $\Phi|_{C'}: C' \rightarrow C$ is a diffeomorphism, we have for any $Z \subset C'$:

$$\text{osc}_Z |\{F, G\}|^p = \text{osc}_{\Phi(Z)} P_{C'}.$$

It then follows that it is enough to prove the following for every $C' \in \mathcal{C}$:

$$\lim_{k \rightarrow \infty} \max_{Q' \in \mathcal{Q}_{n,k}, Q' \subset C'} \text{osc}_{\Phi(Q')} P_{C'} = 0.$$

This follows from the fact that $P_{C'}$ is a smooth function, in particular it has bounded derivatives, and therefore its oscillation over $\Phi(Q')$ is bounded by a constant times the diameter of $\Phi(Q')$ which is $\frac{\sqrt{2}}{kn}$. \square

Proof (of Lemma 2.5). ⁵⁾ Let $X = \text{supp } F \cap \text{supp } G$, $V = \Phi^{-1}(U)$, and $Z = X - V$, which is the subset of X consisting of points lying over singular values of Φ . Let $S, R \subset M$ be the sets of critical and regular points of Φ , respectively. We have the disjoint union ⁶⁾

$$Z = (Z \cap S) \cup (Z \cap R).$$

At the beginning of the proof of Theorem 1.4 we noted that $z \in S$ if and only if $\{F, G\}(z) = 0$, therefore

$$\int_{Z \cap S} |\{F, G\}|^p \omega = 0.$$

We claim that $Z \cap R$ has measure zero. Indeed, $Z \cap R = (\Phi|_R)^{-1}(\text{im } \Phi - U)$, and the claim follows from the fact that R is an open subset of M , therefore a submanifold, $\Phi|_R$ is a local diffeomorphism, the fact that $\text{im } \Phi - U$ has measure zero by Sard's theorem, and the following lemma.

⁵⁾We thank Lev Buhovsky for a suggestion that lead to a simplification of the proof of the lemma.

⁶⁾Note that there may be regular points of Φ which are mapped to singular values.

Lemma 2.6. *Let N, P be manifolds, let $f: N \rightarrow P$ be a local diffeomorphism, and let $Y \subset P$ a subset of measure zero. Then $f^{-1}(Y)$ has measure zero.*

Proof. Since our manifolds are paracompact, they are second countable, and in particular N can be covered with countably many charts, such that on each one of them f is a diffeomorphism onto its image. Since diffeomorphisms preserve the property of having measure zero, it follows that $f^{-1}(Y)$ is covered by countably many measure zero sets, and thus it is itself such. \square

This implies

$$\int_{Z \cap R} |\{F, G\}|^p \omega = 0,$$

and therefore we have

$$\int_M |\{F, G\}|^p \omega = \int_X |\{F, G\}|^p \omega = \int_V |\{F, G\}|^p \omega.$$

From the regularity of the measure $|\{F, G\}|^p \omega$ we obtain

$$\int_V |\{F, G\}|^p \omega = \sup_{K \subset V \text{ compact}} \int_K |\{F, G\}|^p \omega.$$

It is therefore enough to show that for any compact $K \subset V$ there is $n \in \mathbb{N}$ such that $\Phi(K) \subset K_n$. This follows from the fact that $\Phi(K)$ is compact and contained in U , therefore $d(\Phi(K), \mathbb{R}^2 - U) > 0$ and

$$\lim_{n \rightarrow \infty} d(K_n, \mathbb{R}^2 - U) = 0,$$

where d is the Euclidean distance between subsets of \mathbb{R}^2 . This limit is indeed zero since K_n contains all the points at a distance at least $\sqrt{2}/n$ from $\mathbb{R}^2 - U$. \square

We now prove the flexibility result, Theorem 1.7.

Proof (of Theorem 1.7). Let $C \subset M$ be as in the formulation of the theorem and fix $\varepsilon > 0$. We need to construct a Poisson commuting pair $\tilde{F}, \tilde{G} \in C_c^\infty(M)$ supported in C and satisfying

$$\|\tilde{F} - F\|_{C^0} < \varepsilon, \quad \|\tilde{G} - G\|_{L^p}^p < \varepsilon.$$

Fix a Riemannian metric d on M . By a simplex in M we mean the image of an embedding $\Delta \rightarrow M$, where Δ is a closed simplex in \mathbb{R}^{2n} . A triangulation of C is a representation of C as a union of such simplices, where every two simplices intersect only in a common face (which is a simplex of lower dimension). A construction described in [Cai61] produces a finite such

triangulation; moreover given $\delta > 0$, every simplex in this triangulation may be assumed to have diameter $< \delta$ with respect to d .

Since C is compact, F is uniformly continuous on it, that is there exists δ such that if $x, y \in C$ satisfy $d(x, y) < \delta$, then $|F(x) - F(y)| < \delta$. Fix such δ and take a triangulation of C with all the simplices having diameter $< \delta$.

For every simplex Q from the triangulation we fix open subsets with smooth boundary $Q_3 \Subset Q_2 \Subset Q_1 \Subset Q$, satisfying

$$\text{Vol}(Q \setminus Q_3) < \varepsilon \cdot \frac{\text{Vol}(Q)}{\|G\|_{C^0}^q \cdot \text{Vol}(C)}.$$

Here $A \Subset B$ means that the closure of A is contained in the interior of B , and Vol is the volume with respect to ω^n . The condition on the volumes is essential for constructing a suitable \tilde{G} .

Construction of \tilde{F} . Consider a simplex Q with open subsets $Q_2 \Subset Q_1 \Subset Q$ as above. We take an auxiliary smooth function $\varphi: Q \rightarrow [0, 1]$ such that $\varphi|_{Q_2} \equiv 0$ and $\varphi|_{Q \setminus Q_1} \equiv 1$. Fix a point $x_0 \in Q_2$. Define \tilde{F} on Q to be

$$\tilde{F}(x) = \varphi(x)F(x) + (1 - \varphi(x))F(x_0).$$

We see that on Q_2 we have $\tilde{F} \equiv F(x_0)$, while outside Q_1 we have $\tilde{F} \equiv F$. See fig. 2. Next, define \tilde{F} on C by gluing all these partially defined functions. Note that the resulting function is well-defined and smooth. Moreover, since F vanishes near ∂C , it is also true for \tilde{F} . Therefore we can extend \tilde{F} by zero to a smooth function on M with support in C . For any $x \in Q$ we have

$$\begin{aligned} |\tilde{F}(x) - F(x)| &= |\varphi(x)F(x) + (1 - \varphi(x))F(x_0) - F(x)| = \\ &= \underbrace{|1 - \varphi(x)|}_{\leq 1} \cdot \underbrace{|F(x) - F(x_0)|}_{< \varepsilon} < \varepsilon, \end{aligned}$$

where the last inequality holds since $\text{diam}(Q) < \delta$. Since Q is arbitrary, we obtain $\|\tilde{F} - F\|_{C^0} < \varepsilon$.

Construction of \tilde{G} . Consider again a simplex Q from our triangulation with the subsets $Q_3 \Subset Q_2 \Subset Q_1 \Subset Q$, such that

$$\text{Vol}(Q \setminus Q_3) \leq \varepsilon \cdot \frac{\text{Vol}(Q)}{\|G\|_{C^0}^q \cdot \text{Vol}(C)}.$$

Take a smooth function $\psi: Q \rightarrow [0, 1]$ satisfying $\psi|_{Q_3} \equiv 1$, $\psi|_{Q \setminus Q_2} \equiv 0$, and define $\tilde{G}: Q \rightarrow \mathbb{R}$ by $\tilde{G} = \psi G$. We have $\tilde{G} \equiv G$ on Q_3 and $\tilde{G}|_{Q \setminus Q_2} \equiv 0$. Take \tilde{G} to be the function on M defined in this way on every simplex Q , and extended by zero to $M \setminus C$. This again is a well-defined smooth function with support in C .

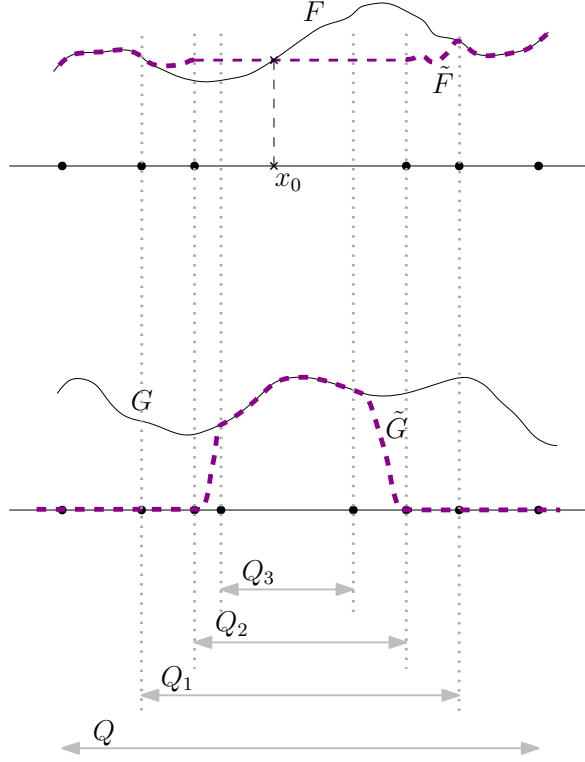


Figure 2: Producing \tilde{F} and \tilde{G} (the dashed lines).

On a single simplex Q we have

$$\begin{aligned}
\int_Q |\tilde{G} - G|^q \omega^n &= \int_{Q \setminus Q_3} |\tilde{G} - G|^q \omega^n \\
&= \int_{Q \setminus Q_3} (1 - \psi)^q |G|^q \omega^n \\
&\leq \int_{Q \setminus Q_3} |G|^q \omega^n \\
&\leq \|G\|_{C^0}^q \cdot \text{Vol}(Q \setminus Q_3) \\
&< \|G\|_{C^0}^q \cdot \varepsilon \cdot \frac{\text{Vol}(Q)}{\|G\|_{C^0}^q \text{Vol}(C)} = \varepsilon \cdot \frac{\text{Vol}(Q)}{\text{Vol}(C)}.
\end{aligned}$$

Therefore on the whole of M we get the bound

$$\|\tilde{G} - G\|_{L^q(M)}^q = \int_M |\tilde{G} - G|^q \omega^n < \varepsilon.$$

It remains to note that for every simplex Q in our triangulation and the associated subsets $Q_3 \subseteq Q_2 \subseteq Q_1 \subseteq Q$, \tilde{F} is constant on Q_2 , while $\tilde{G} \equiv 0$ outside Q_2 , meaning $\{\tilde{F}, \tilde{G}\} \equiv 0$ as claimed. \square

References

- [Buh10] Lev Buhovsky. The $2/3$ -convergence rate for the Poisson bracket. *Geom. Funct. Anal.*, 19(6):1620–1649, 2010.
- [Cai61] Stewart S. Cairns. A simple triangulation method for smooth manifolds. *Bull. Amer. Math. Soc.*, 67:389–390, 1961.
- [CV08] Franco Cardin and Claude Viterbo. Commuting Hamiltonians and Hamilton–Jacobi multi-time equations. *Duke Math. J.*, 144(2):235–284, 2008.
- [EP10] Michael Entov and Leonid Polterovich. C^0 -rigidity of Poisson brackets. In *Symplectic topology and measure preserving dynamical systems*, volume 512 of *Contemp. Math.*, pages 25–32. Amer. Math. Soc., Providence, RI, 2010.
- [EPZ07] Michael Entov, Leonid Polterovich, and Frol Zapolsky. Quasimorphisms and the Poisson bracket. *Pure Appl. Math. Q.*, 3(4, Special Issue: In honor of Grigory Margulis. Part 1):1037–1055, 2007.
- [Fed69] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.
- [Sam15] Karina Samvelyan. Rigidity versus flexibility of the Poisson bracket with respect to the L_p -norm. Master’s thesis, Tel Aviv University, 2015.
- [Zap07] Frol Zapolsky. Quasi-states and the Poisson bracket on surfaces. *J. Mod. Dyn.*, 1(3):465–475, 2007.